1.2 The display of polynomials

Consider the polynomial $f = 4\,x^4 + 4\,x^3 - 13\,x^2 - 7\,x + 8$ *!*

a) Determine all the real roots of the f!

b) Plot the f in an interval which contains all the real roots. Try to plot a graph that looks fine!

c) Determine the equation of the tangent of the f at the x=0 point and plot the f and its tangent in the same graph!

We have met some services of the **2-D Math** input. We have learnt that it is not always the typed characters that can be seen on the screen in this input mode. For example, when we press the sign of multiplication *, the system substitutes it with a dot (.). The difference is much more striking in the case of the input of fractions when the system converts the sequentially typed characters into a two dimensional formula.

While the sign of division (/) operates as a kind of control character, underscore (_) is responsible for the subscripts. But right now the aim is to input a polynomial into the system so the task is to generate the exponent. In **2-D Math**, the character ^ makes the cursor switch into the superscript. If we use a Hungarian keyboard, press **ALT GR+3** to get it. This procedure is annoying in itself and what is more, after pressing **ALT GR+3** the character will only be displayed if we press the subsequent character as well. This is rather awful but get used to it. We can exit superscript with the **right arrow** cursor control key (supposing that the cursor is blinking after the last character of the superscript).

To practice, enter the formula x^4. The characters to be typed are: **x**, **ALT GR + 3**, **4** és **Enter**.

> x^4

The next exercise is to enter $x^4 + x^3$. In the course of the previous command, the cursor was located in the superscript when we closed the command with **Enter**. Now, however, we have to exit it in order to type the second half of the formula. The characters to be typed are: **x**, **ALT GR + 3**, **4**, **right arrow**, **+**, **x**, **ALT GR + 3**, **3**, **Enter**.

> $x^4 + x^3$

The entering of * is cumbersome in itself if we use a keyboard without a numeric keypad. Usually we have to press **ALTGR+ -** to get it. That is why it is very convenient that **2-D Math** accepts (**SPACE**) as the sign of multiplication. Let's try it. Input the expression $4\,x^4$ by typing the following characters: **4, SPACE, x, ALT GR + 3, 4** and **Enter**

> $4\,x^4$

After such a long preparation it is time for us to start the task. Enter the polynomial f!

> $f := 4\,x^4 + 4\,x^3 - 13\,x^2 - 7\,x + 8$

$$f := 4\,x^4 + 4\,x^3 - 13\,x^2 - 7\,x + 8 \qquad \textbf{(1)}$$

If everything has gone smoothly, we can see the line above on the screen.

When we start to use the system we can easily forget the parameters of the different procedures. Can we tell the parameters of the **solve** procedure mentioned in the previous chapter by heart? If not, the **example** procedure helps us with illustrations of the call of the procedure given as a parameter. The *example*(*solve*) command opens a new window on the screen where we can see examples of the possible calls of the **solve** procedure.

> *example*(*solve*)   *# The command opens a new window on the screen.*

Notice that by starting with the # symbol we typed a comment in the command line above. These comments can greatly help the understanding of the worksheets.

The first parameter of the **solve** procedure is an equation and the second parameter is the variable according to which the equation is to be solved. The examples also show that if the **solve** gets an f expression as its first parameter then the system considers the f=0 as the equation to be solved, that is, when we call the **solve** procedure, the =0 sign may be omitted.

In order to solve our task, not only the roots of the polynomial f are needed but the smallest and the largest root as well. Use the **min** and **max** procedures of Maple to get them.

> *solve*(*f*, *x*)

$$-\frac{1}{4}-\frac{1}{4}\sqrt{29-4\sqrt{17}},\ -\frac{1}{4}+\frac{1}{4}\sqrt{29-4\sqrt{17}},\ -\frac{1}{4}-\frac{1}{4}\sqrt{29+4\sqrt{17}},\ -\frac{1}{4}+\frac{1}{4}\sqrt{29+4\sqrt{17}} \qquad \textbf{(2)}$$

> *maximum* := max(%)

$$maximum := -\frac{1}{4}+\frac{1}{4}\sqrt{29+4\sqrt{17}} \qquad \textbf{(3)}$$

> *minimum* := min(%%)

$$minimum := -\frac{1}{4}-\frac{1}{4}\sqrt{29+4\sqrt{17}} \qquad \textbf{(4)}$$

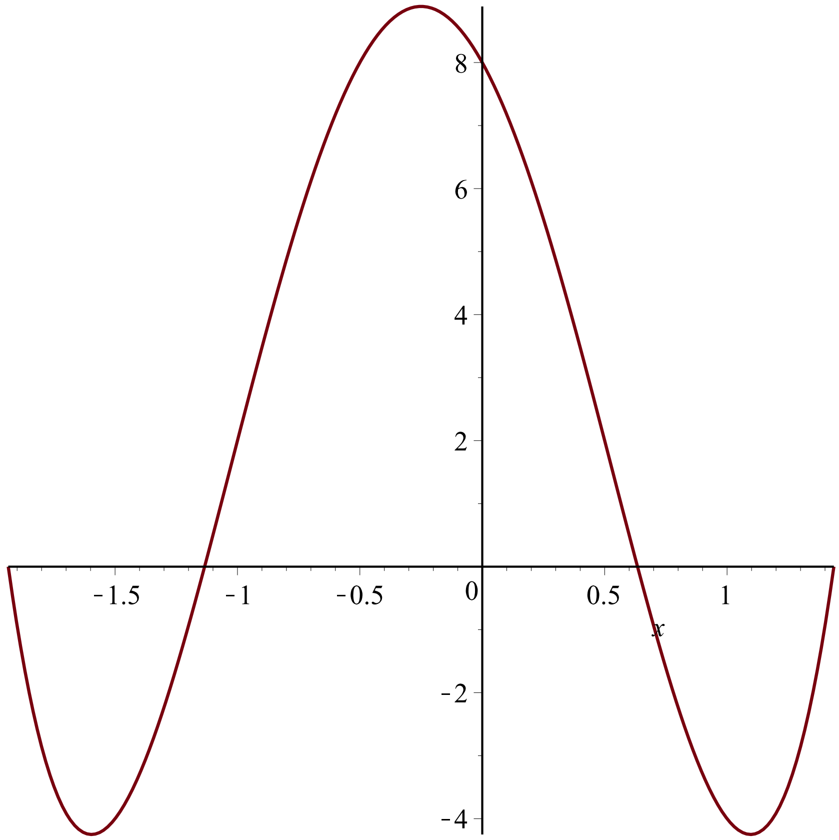
>

The **solve** procedure has found four real roots out of which we probably cannot decide which is the smallest and the largest. The **max** procedure – for which we gave the % sign as a parameter – will come in handy. The percent sign represents the result of the previously executed instruction in Maple. In this case **max** chooses the largest root out of the four roots generated by **solve**. We gave two percentage signs (%%) to the min procedure as a parameter. Obviously, this represents the result of the operation prior to the previously executed operation. The question is: how many percentage signs can be put sequentially? Only three, that is, we can refer to the results of the previous operations like this: %, %% and %%%. There is no need to have more percentage signs because who could remember the results of the four times or even more times earlier executed commands?

So why do we need the reference of the last three executed operations when we have the labels of the outputs? It is true that when choosing the largest and the smallest elements we could have directly referred to the output (3). But labels can be disabled (in **Tools** menu **Options**) and then the percentage sign reference remains the only one available. On the other hand, it is beneficial if we can choose between the absolute reference guaranteed by the labels and the relative reference executed by the percentage signs.

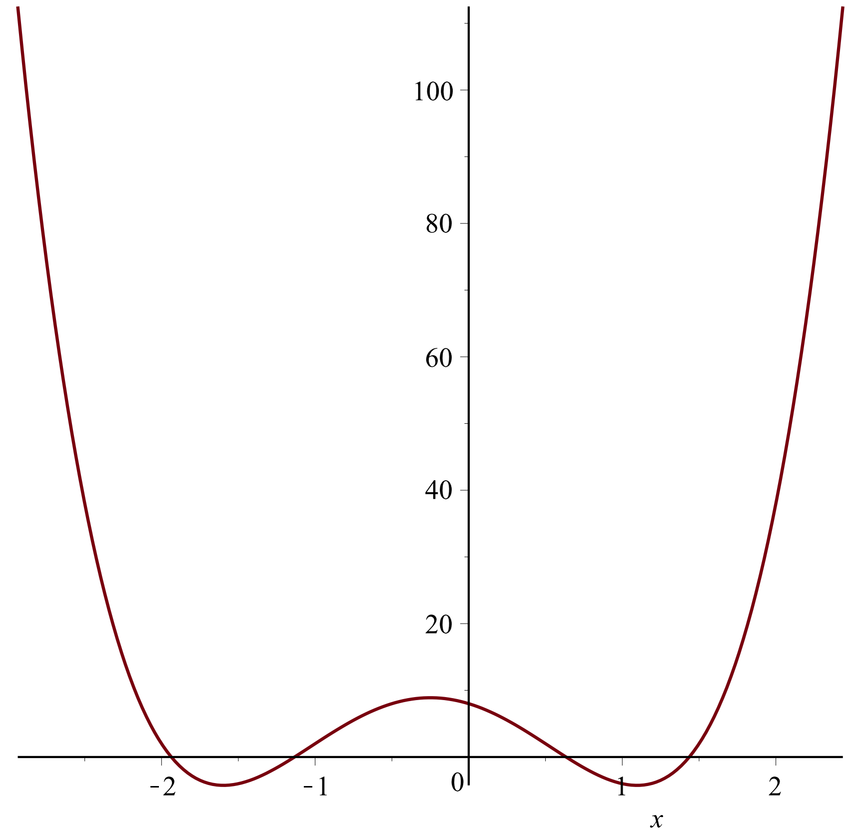
We have determined the smallest and the largest real roots of the polynomial *f*. In accordance with the task we have to plot the polynomial in the interval the left endpoint of which is the value of the *minimum* variable and the right endpoint is the value of the *maximum* variable. The general drawing function of Maple is called **plot** the first parameter of which is the expression to be plotted and the second parameter is the domain on the *x*–axis.

> *plot*(*f*, *x* = *minimum* ..*maximum*)



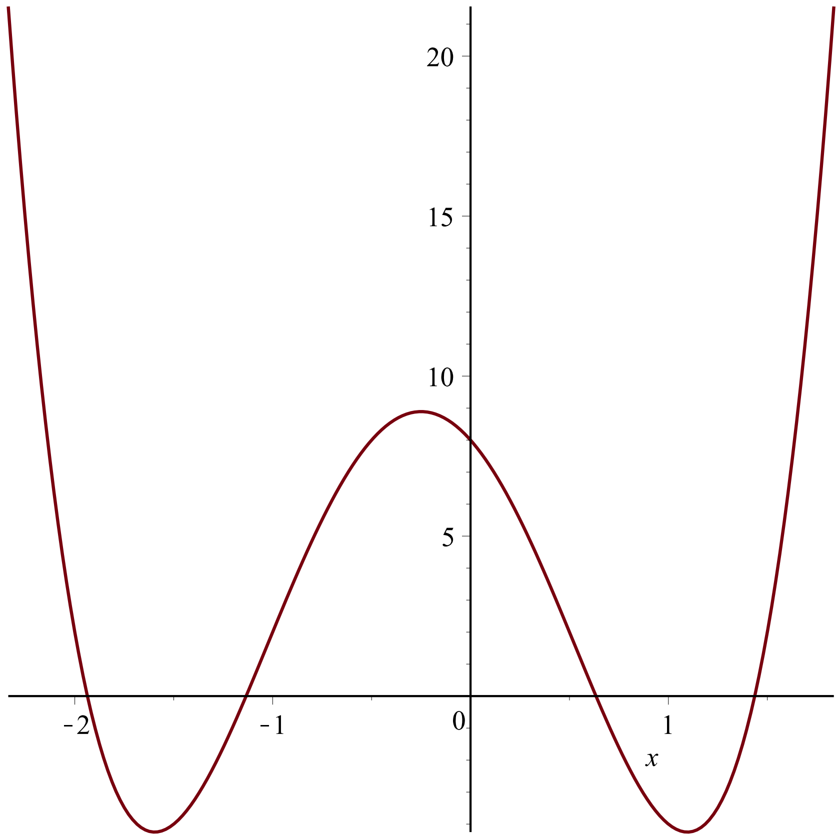
Are we satisfied with the displayed graph? The function itself looks fine but it is a bit awkward that the curve of the function ends in the intersection point shared with the *x* axis. As a solution we should give a wider interval.

> $plot(f, x = minimum - 1\ ..maximum + 1)$



The middle arc of the function has shrunk. We can wonder why. Notice that while in the first graph approximately the [-4,8] interval, in the second graph the [-4,110] interval is the domain represented on the y–axis. Maple calculates the smallest and the largest value of the function in a given domain and it scales the y–axis based on this. Accordingly, the value of the function is too big in the $minimum - 1$ and $maximum + 1$ endpoints. Let's try the following command.

> $plot(f, x = minimum - 0.4 .. maximum + 0.4)$



What a nice graph. Keep on experimenting.

Now let's continue the solution of the task. To determine the tangent of the f we need the slope of the tangent in the $(0, f(0))$ point. As we know the tangent is the value of the derivative of the f. Therefore, we have to calculate the derivative of the f. Should we not remember how to call the **diff** procedure, use the **example** procedure!

> *example*(*diff*)   *# omit the output of the command*

> *derivált* := *diff*(*f*, *x*)

$$\mathit{derivált} := 16\,x^3 + 12\,x^2 - 26\,x - 7 \tag{5}$$

> $x_0 := 0$

$$x_0 := 0 \tag{6}$$

> $y_0 := eval(f, x = x_0)$

$$y_0 := 8 \tag{7}$$

> $g := diff(f, x)$

$$g := 16\,x^3 + 12\,x^2 - 26\,x - 7 \tag{8}$$

> $m := eval(g, x = x_0)$

$$m := -7 \tag{9}$$

> $y := y_0 + m \cdot (x - x_0)$

$$y := 8 - 7\,x \tag{10}$$

>

After this we have to execute the $x = 0$ substitution in the derivative. The **subs** procedure is used for this.

> $t := subs(x = 0, \mathit{derivált})$

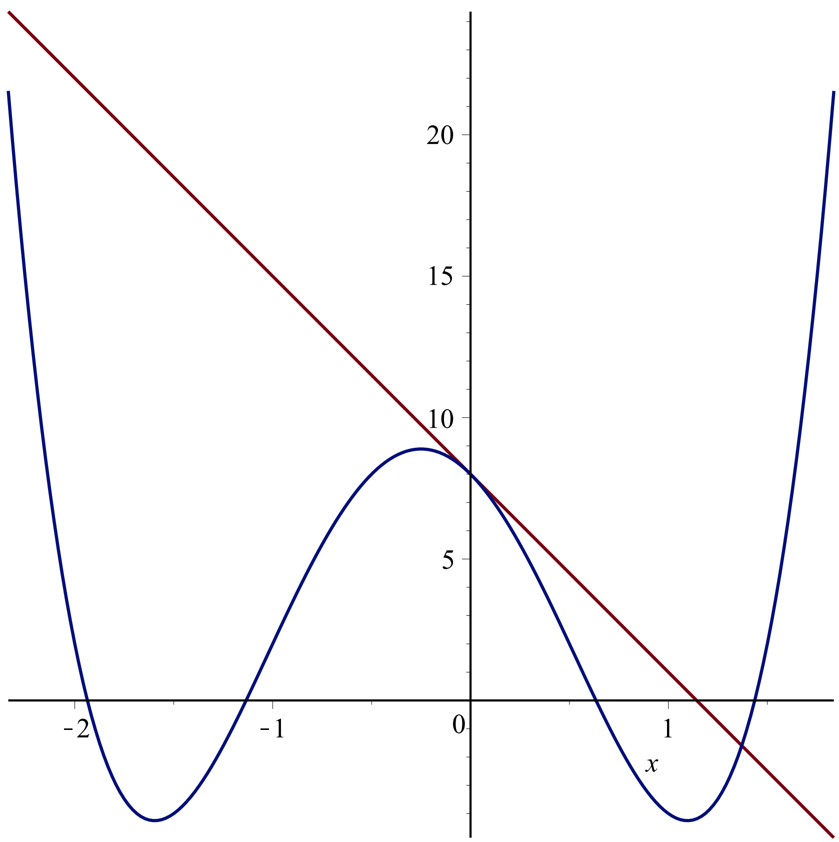
$$t := -7 \tag{11}$$

As the result of this command, Maple substitutes all the appearances of the variable x in the f with zero while it leaves the derivative unchanged in the variable name *derivált*. After this the equation of the tangent line and the desired graph can be plotted with the following commands.

> $y := subs(x = 0, f) + t\,(x + 0)$

$$y := 8 - 7\,x \tag{12}$$

> $plot(\{f, y\}, x = minimum - 0.4 .. maximum + 0.4)$



>

With the first command we entered the equation of the tangent line with slope t which goes through the ($0, f(0)$) point. More precisely the value of the variable y is the expression that describes the line. We gave the two-element set $\{f, y\}$ as the first parameter of the **plot** command which made **plot** display the two curves in one graph.

Syntactically **set** – which is another example of Maple data types – is the sequence of Maple objects between curly brackets. Usually in mathematics a set is the unsorted collection of its elements. The sign of an empty set is {} and the system can execute common set operations such as **union**, **intersect** and **minus**. Let's take some examples!

Enter the sets H_1 and H_2. Note that the elements of the sets might be different objects, numbers, names and expressions. Remember that we have to type the underscore character (_) in **2-D Math** to switch to subscript mode.

> $H_1 := \{1, sin(x), x^2 - 4\}; H_2 := \{Maple, sin(x)\}$

$$H_1 := \{1, x^2 - 4, \sin(x)\}$$
$$H_2 := \{Maple, \sin(x)\} \qquad \textbf{(13)}$$

The following commands do not need further explanation.

> H_1 *intersect* H_2

$$\{\sin(x)\} \qquad \textbf{(14)}$$

> H_1 *union* H_2

$$\{1, Maple, x^2 - 4, \sin(x)\} \qquad \textbf{(15)}$$

> H_1 *minus* H_2

$$\{1, x^2 - 4\} \qquad \textbf{(16)}$$

> H_1 *subset* H_2

$$false \qquad \textbf{(17)}$$

During the **2-D Math** input the aim was to make the input resemble the common mathematical signs as much as possible. This is not true for the set theoretical operations previously mentioned. The solution is to use the palettes of the system.

Palettes are charts which contain useful characters out of which the desired character can be placed on the worksheet by clicking or with drag and drop.

Maple provides a number of palettes for the users. These palettes can be found on the so-called docks situated on the two sides of the worksheet. Docks can be shown and hidden. Docks can be made visible with the **Expand Docks** command of the **Palettes** submenu of the **View** menu and can be made disappear with the **Collapse Docks** command.

The palettes of the docks can be opened or closed. We can open and close them by clicking on the triangle

standing in front of the name of the palette.

The **Common Symbol** palette, which is displayed in the figure as open, contains a lot of common mathematical symbols.

The previous operations seem easier if we use the set theory operator signs of the **Common Symbol** palette. Here are the keys to be pressed at the input of the first command: **H, _, 1, Rightarrow, `Click on ∩ symbol of the palette`, H, _, 2, Enter.**

> $H_1 \cap H_2$

$$\{\sin(x)\} \tag{18}$$

> $H_1 \cup H_2$

$$\{1, \mathit{Maple}, x^2 - 4, \sin(x)\} \tag{19}$$

> $H_1 \setminus H_2$

$$\{1, x^2 - 4\} \tag{20}$$

> $H_1 \subseteq H_2$

$$\mathit{false} \tag{21}$$

>

What have we learnt about Maple?

- If we press **ALTGR+3** during the input of the mathematical signs in **2-D Math** the cursor will switch into superscript. Here we can give the exponent and exit with the **right arrow key**. More precisely the **right arrow key** makes the cursor exit the superscript.
- The formats of the inputs can be improved by using the palettes. We can find several common mathematical signs in the **Common Symbols** palette. These signs can be placed in the relevant cursor position of the worksheet by clicking on the sign.
- The **example** procedure illustrates how to call the procedure which has been given as a parameter, e.g. *example*(*solve*), *example*(*diff*), etc.

- The **solve** procedure accepts expressions instead of equations and in this case it solves the $expression = 0$ equation.
- While the **max** procedure chooses the largest, the **min** procedure chooses the smallest element of the sequence given as their parameter.
- The easiest call of the **plot** procedure is $plot(expression, x = a\,..b)$. In this case the system displays the curve defined by the expression in the $[a, b]$ closed interval.
- If we give the set of expressions to the **plot** procedure then the system displays the curves in a graph with different colours.
- The **subs** procedure creates the substitution value of the expressions. The simplest way to do this is: $subs\left(variable = expression_1, expression_2\right)$. It makes all the appearances of the *variable* in each $expression_2$ be substituted by the value of the $expression_1$. Notice that the substitution does not change the $expression_2$!
- Data type set is the sequence of Maple objects between curly brackets. A set is the unsorted collection of its elements. Set operations are union, intersect and minus.
- The **Common Symbols** palette contains the common signs of the set operations.

Exercises

1. Practice the commands by typing the following input lines. Try to figure out the responses of the system in advance!

- $2 + 3$
- 28^2
- $(a + b) \cdot (a - b)$
- $y = a\,x^2 + b\,x + c$
- $f := e^{\left(-\frac{1}{x^2}\right)}$
- $g := \cos\left(\frac{x^2+1}{x}\right) \sin\left(\frac{x}{x^2-1}\right)$
- $\sin\left(\frac{x}{4}\right)$
- 123456787654321
- $\frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}\ \frac{4}{5}$
- $s := \mathrm{MAPLE}, \tan(x), \frac{3}{7}$
- $s := 1, 2, 3, 4$
- $s := \frac{\tan(x)}{\cos(x)}, \tan(x) \cdot \cos(x)$

2. Determine the derivatives of the following functions.

- $x^{3-23}\,x + 9$
- $x\,(x + 1)\,\left(x^2 + 1\right)$
- $\left(x^{3-1}\right)^2$
- $\sin(x) + \cos(x) + \tan(x)$

- $\ln(\tan(x))$
- $\sqrt{\arctan\left(\sqrt{\frac{1-x}{1+x}}\right)}$

3. Find the roots of the following equations.
    - $a x^2 + b x + c = 0;$
    - $a x^3 + b x^2 + c x + d = 0$
    - $x^4 + x^3 + 4 x^2 - 4 = 0$
    - $x^3 - 2 \cdot x^2 - 30 = 0$
    - $3 x^3 + 2 x^2 + x + 1 = 0$

4. Prove that if the five real roots of a fifth degree polynomial form an arithmetical sequence then the same is true for the roots of its second derivative.

5. Consider the function $f(x) = x^3 - 3 x^2$ and the fourth degree polynomial $y(x) = x f(x - 1)$ that derives from it. Prove that the roots of the derivatives of the y form a geometric sequence.